E9 203 Compressed Sensing & Sparse SP

30 Apr. 2021.

Last time:
- Restricted Isometry Constant (RIC)
- Restricted Orthogonality constant (ROC).
- Some properties

Today:
- Further properties & bounds on RIC

Recap:

Defn. $A \in \mathbb{C}^{m\times N}$, s^{th} RIC $\delta_s = \delta_s(A)$ is the smallest $\delta \geq 0$ s.t.

$\left(\left| \|Ax\|_2^2 - \|x\|_2^2 \right| \geq \delta \|x\|_2^2 \right)$

$$\left\{ (1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2 \right. \quad \text{— (1)}$$
for all s-sparse $x \in \mathbb{C}^N$.

Equivalently,
$$\delta_s = \max_{\substack{S \subset [N] \\ |S| \leq s}} \| \underline{A_S^H A_S} - I \|_{2\to 2} \quad \text{— (2)}$$

Prop. 6.2: $A \in \mathbb{C}^{m\times N}$ ℓ_2 normalized cols

$\delta_1 = 0$, $\delta_2 = \mu_1$, $\underline{\delta_s \leq \mu_1(s-1)} \leq (s-1)\mu$. — (3)
$(s \geq 2)$.

Prop. 6.3 $u, v \in \mathbb{C}^N$, $\|u\|_0 \leq s$, $\|v\|_0 \leq t$

If $\text{supp}(u) \cap \text{supp}(v) = \emptyset$, then
$$|\langle Au, Av \rangle| \leq \delta_{s+t} \|u\|_2 \|v\|_2 . \quad \text{— (4)}$$

Defn. The (s,t)-restricted orthogonality const. $\theta_{s,t} = \theta_{s,t}(A)$ of $A \in \mathbb{C}^{m\times N}$ is the smallest $\theta \geq 0$ s.t.
$$|\langle Au, Av \rangle| \leq \theta \|u\|_2 \|v\|_2$$
for all disjointly supported s- and t-sparse $u, v \in \mathbb{C}^N$.

Equivalently, $\theta_{s,t} = \max \{ \|A_T^H A_S\|_{2\to 2},\ S \cap T = \emptyset,\ |S| \leq s,\ |T| \leq t \}$. — (5)

Prop. 6.5 The RIC and ROC are related by
$$\underbrace{\theta_{s,t} \leq \delta_{s+t}}_{\text{Prop. 6.3}} \underbrace{\leq \frac{1}{s+t}}_{\text{To show.}} (s\delta_s + t\delta_t + 2\sqrt{st}\,\theta_{s,t})$$

In the spl. case $t = s$, $\underline{\theta_{s,s} \leq \delta_{2s} \leq \delta_s + \theta_{s,s}}$.
direct from substitution.

Proof: Consider an $(s+t)$-sparse $x \in \mathbb{C}^N$, $\|x\|_2 = 1$.

Need to show that:
$$\underbrace{\left| \|Ax\|_2^2 - \|x\|_2^2 \right|}_{\leq \delta_{s+t}\ \forall (s+t) \text{ sparse } x \in \mathbb{C}^N} \leq \frac{1}{s+t}(s\delta_s + t\delta_t + 2\sqrt{st}\,\theta_{s,t})$$

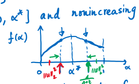
Let $u, v \in \mathbb{C}^N$ be disjointly supp. vecs. s.t. $u + v = x$, u is s-sparse, v is t-sparse.

$$\|Ax\|_2^2 = \langle A(u+v), A(u+v) \rangle$$
$$= \|Au\|_2^2 + \|Av\|_2^2 + 2\,\text{Re}\,\langle Au, Av \rangle. \quad \text{— }\theta$$

Since $1 = \|x\|_2^2 = \|u\|_2^2 + \|v\|_2^2$,
$$\left| \|Ax\|_2^2 - \|x\|_2^2 \right| \leq \left| \|Au\|_2^2 - \|u\|_2^2 \right| + \left| \|Av\|_2^2 - \|v\|_2^2 \right| + 2 |\langle Au, Av \rangle|$$
$$\leq \underbrace{\delta_s \|u\|_2^2 + \delta_t \|v\|_2^2 + 2\theta_{s,t}\|u\|_2\|v\|_2}_{\triangleq f(\underbrace{\|u\|_2^2}_{\alpha})}$$

Here, for $\alpha \in [0,1]$
$$f(\alpha) \triangleq \delta_s \alpha + \delta_t (1-\alpha) + 2\theta_{s,t}\sqrt{\alpha(1-\alpha)}.$$

[HW]: $\exists\, \alpha^* \in [0,1]$ s.t. $f(\alpha)$ is nondecreasing on $[0, \alpha^*]$ and nonincreasing on $[\alpha^*, 1]$.



$0 \leq \left(\frac{s}{s+t}\right) \leq 1$

Depending on the location of $\frac{s}{s+t}$ w.r.t. α^*, $f(\alpha)$ is either nondecreasing on $[0, \frac{s}{s+t}]$ or nonincreasing on $[\frac{s}{s+t}, 1]$.

Can choose u s.t. $\|u\|_2^2$ always lies in one of these intervals. If u is made of the s smallest abs. entries of x (v is made of the t largest abs. entries of x), then
$$\frac{\|u\|_2^2}{s} \leq \frac{\|v\|_2^2}{t} = \frac{1 - \|u\|_2^2}{t} \Rightarrow \underline{\|u\|_2^2 \leq \frac{s}{s+t}}.$$

If u is made of the s largest abs. entries of x (v is made of the t smallest abs. entries of x), then
$$\|u\|_2^2 \geq \frac{s}{s+t}.$$

This $\Rightarrow \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \leq f\left(\frac{s}{s+t}\right)$
$$= \delta_s\left(\frac{s}{s+t}\right) + \delta_t\left(\frac{t}{s+t}\right) + 2\theta_{s,t}\frac{\sqrt{st}}{s+t}$$
$$= \frac{1}{s+t}(s\delta_s + t\delta_t + 2\theta_{s,t}\sqrt{st}). \quad \square$$

Thm. 6.8 $A \in \mathbb{C}^{m\times N}$, $\underline{2 \leq s \leq N}$, then
$$m \geq c\,\frac{s}{\delta_s^2}$$
provided $N \geq Cm$ and $\delta_s \leq \delta_*$, where c, C and δ_* depend only on each other.

Ex. $c = \frac{1}{162}$, $C = 30$, $\delta_* = \frac{2}{3}$ is valid.

Proof: The statement cannot hold for $s = 1$, $\because \delta_1 = 0$ if all cols. of A have unit ℓ_2 norm.

Let $t = \left\lfloor \frac{s}{2} \right\rfloor \geq 1$

Partition A into blocks of size $m \times t$
$$A = [A_1 \,\vdots\, A_2 \,\vdots\, \cdots \,\vdots\, A_n], \quad N \leq nt$$

From the defns. of RIC & ROC (see (2), (5)):
$\forall\, i, j \in [n]$, $i \neq j$,
$$\begin{cases} \|A_i^H A_i - I\|_{2\to 2} \leq \delta_t \leq \delta_s \\ \|A_i^H A_j\|_{2\to 2} \leq \theta_{t,t} \leq \delta_{2t} \leq \delta_s . \end{cases}$$

$[A_i^H A_j] \in \mathbb{C}^{N\times N}$

Define $H = AA^H \in \mathbb{C}^{m\times m}$, $G = A^H A = [A_i^H A_j]_{1\le i,j\le n}$

Lower bound:

$$\mathrm{tr}(H) = \mathrm{tr}(G) = \sum_{i=1}^{n} \mathrm{tr}(A_i^H A_i) = \sum_{i=1}^{n}\sum_{k=1}^{t} \lambda_k(A_i^H A_i)$$
$$\geq nt(1-\delta_s)$$

Consider $\langle M_1, M_2\rangle_F = \mathrm{tr}(M_2^* M_1)$ [Frobenius inner prod.]

$$\mathrm{tr}(H)^2 = |\langle I, H\rangle_F|^2 \leq \|I\|_F^2 \, \|H\|_F^2 = m\,\mathrm{tr}(H^H H)$$

By the cyclic prop. of trace,

$$\mathrm{tr}(H^H H) = \mathrm{tr}(AA^HAA^H) = \mathrm{tr}(A^HAA^HA) = \mathrm{tr}(GG^H)$$
$$= \sum_{i=1}^{n} \mathrm{tr}\Big(\sum_{j=1}^{n} A_i^H A_j A_j^H A_i\Big)$$
$$= \sum_{1\le i\ne j\le n} \sum_{k=1}^{t} \sigma_k(A_i^H A_j)^2 + \sum_{i=1}^{n}\sum_{k=1}^{t} \lambda_k(A_i^H A_i)^2$$
$$\leq n(n-1)t\,\delta_s^2 + nt(1+\delta_s)^2$$

$$\mathrm{tr}(H)^2 \leq mnt\big((n-1)\delta_s^2 + (1+\delta_s)^2\big)$$
$$n^2t^2(1-\delta_s)^2 \leq mnt\big((n-1)\delta_s^2 + (1+\delta_s)^2\big)$$

$$\Rightarrow m \geq \frac{nt(1-\delta_s)^2}{\underbrace{(n-1)\delta_s^2}_{} + (1+\delta_s)^2}.$$

Can s.t. this is the bigger of the two terms in the denominator.

Replacing $(1+\delta_s)^2$ by $5(n-1)\delta_s^2$ (an upper bd.)

$$\Rightarrow m \geq \frac{nt(1-\delta_s)^2}{6(n-1)\delta_s^2} \geq \frac{1}{54}\frac{t}{\delta_s^2} \geq \frac{1}{162}\frac{s}{\delta_s^2}$$

$\because \delta_s < \frac{2}{3}$ $\quad$ $s < 3t$ $\quad$ ☒